تمرین ۲

بخش ۱

فرض کنید $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ و $A \in M_{n,n}(\mathbb{C})$ باشند. کدام یک از عبارات زیر امکان‌پذیر هستند؟ با شرح مناسب برای هر گزینه مشخص کنید.

(a) $|\psi\rangle + \langle\phi|$

(b) $|\psi\rangle\langle\phi|$

(c) $A\langle\psi|$

(d) $|\psi\rangle A$

(e) $\langle\psi|A$

(f) $\langle\psi|A + \langle\phi|$

(g) $|\psi\rangle|\phi\rangle$

(h) $|\psi\rangle\langle\phi|A$

(i) $|\psi\rangle A\langle\phi|$

(j) $\langle\psi|A|\phi\rangle$

(k) $\langle\psi|A|\phi\rangle + \langle\psi|\phi\rangle$

(l) $\langle\psi|\phi\rangle\langle\psi|$

(m) $\langle\psi|\phi\rangle A$

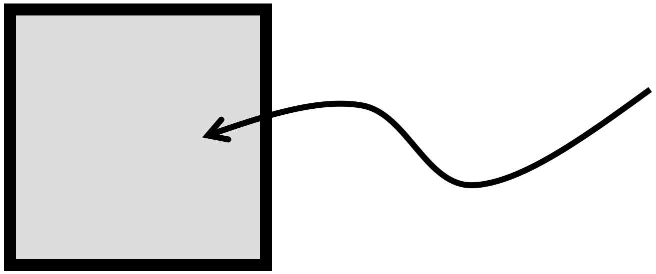
(n) $|\psi\rangle\langle\psi|\phi\rangle = \langle\psi|\phi\rangle|\psi\rangle$

بخش ۲

تمام «مثال‌هایی» که در ادامه در باکس‌های خاکستری رنگ آمده است را به صورت مختصر توضیح دهید.

مثلاً: مثال صفحه ۳ یک بردار احتمال است که احتمال حضور در سه حالت را نشان می‌دهد.

Rule 1



Machine has N states

0,1,2,…,N-1

N dimensional real vector

$$p \in \mathbb{R}^N \qquad p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-1} \end{bmatrix}$$

positive elements

$$p_i \geq 0$$

which sum to unity

$$\sum_{i=0}^{N-1} p_i = 1$$

probability vector

Example: 3 state device

$$p = \begin{bmatrix} 0.3 \\ 0.7 \\ 0 \end{bmatrix}$$

30 % state 0
70 % state 1
0 % state 2

Rule 2

$$q_j = \sum_{i=0}^{N-1} A_{j,i} p_i$$

these are probabilities

$$A_{j,i} \geq 0$$

stochastic matrix

$$\sum_{j=0}^{N-1} A_{j,i} = 1$$

If in state 0 switch to state 0 with probability 0.4

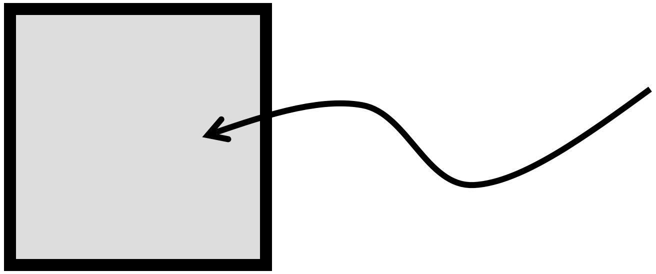
If in state 0 switch to state 1 with probability 0.6

If in state 1 always stay in state 1

$$A = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 1 \end{bmatrix} \quad p = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \quad \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \rightarrow \begin{bmatrix} 0.16 \\ 0.82 \end{bmatrix}$$

$$q = Ap = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 1 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 0.84 \end{bmatrix}$$

Quantum Rule 1



Machine has N states

0,1,2,…,N-1

N dimensional complex vector (vector of amplitudes)

$$v \in \mathbb{C}^N \qquad v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \qquad v_i \in \mathbb{C}$$

Example: 2 state device

$$v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$\sum_{i=0}^{N-1} |v_i|^2 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\frac{i}{\sqrt{2}} = 1$$

Quantum Rule 2, Example

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

Conjugate: $$U^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}$$

Conjugate transpose: $$U^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}$$

Unitary?

$$UU^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

evolves to

$$v' = Uv = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Quantum Rule 1, Probabilities?

If we measure our quantum information processing machine, (in the computational basis) when our description is v, then the probability of observing state i is $|v_i|^2$

$$v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

quantum state

$$\begin{bmatrix} |v_0|^2 \\ |v_1|^2 \\ \vdots \\ |v_{N-1}|^2 \end{bmatrix} \quad \begin{matrix} Pr(0) \\ Pr(1) \\ \\ Pr(N-1) \end{matrix}$$

probabilities

requirement of unit vector insures these are probabilities

Example:

$$v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$Pr(0) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2} \qquad v' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Pr(1) = \frac{-i}{\sqrt{2}}\frac{i}{\sqrt{2}} = \frac{1}{2} \qquad v' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Computational Basis

Some special vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad |N-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example:

2 dimensional complex vectors (also known as: a qubit!)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Computational Basis

Vectors can be "expanded" in the computational basis:

$$|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} = v_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + v_{N-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle$$

Example:

$$|v\rangle = \begin{bmatrix} 1+2i \\ 3 \end{bmatrix} = (1+2i)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (1+2i)|0\rangle + 3|1\rangle$$

Computational Bras

Computational Basis, but now for bras:

$$\langle 0| = \begin{bmatrix} 1 & 0 & \cdots 0 \end{bmatrix}$$

$$\langle 1| = \begin{bmatrix} 0 & 1 & \cdots 0 \end{bmatrix}$$

$$\vdots$$

$$\langle N-1| = \begin{bmatrix} 0 & 0 & \cdots 1 \end{bmatrix}$$

$$\langle v| = \begin{bmatrix} v_0^* & v_1^* & \cdots v_{N-1}^* \end{bmatrix} = v_0^*\langle 0| + v_1^*\langle 1| + \cdots + v_{N-1}^*\langle N-1|$$

Example:

$$\langle v| = \begin{bmatrix} 2 & 3+2i \end{bmatrix} = 2\langle 0| + (3+2i)\langle 1|$$

The Inner Product

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

Example:

$$|v\rangle = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} \qquad |w\rangle = \begin{bmatrix} 3i \\ 3 \end{bmatrix}$$

$$\langle w| = \begin{bmatrix} -3i & 3 \end{bmatrix}$$

$$\langle w|v\rangle = \begin{bmatrix} -3i & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} = (-3i)\cdot 1 + 3(1+2i) = 3+3i$$

$$\langle v|w\rangle = \begin{bmatrix} 1 & 1-2i \end{bmatrix} \begin{bmatrix} 3i \\ 3 \end{bmatrix} = 1\cdot(3i) + (1-2i)3 = 3-3i$$

Complex conjugate of inner product: $(\langle w|v\rangle)^* = \langle v|w\rangle$

The Inner Product in Comp. Basis

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

$$\langle w| = w_0^*\langle 0| + w_1^*\langle 1| + \cdots + w_{N-1}^*\langle N-1|$$

$$|v\rangle = v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle$$

$|0\rangle\langle 0|$ NO!

$$\langle w|v\rangle = (w_0^*\langle 0| + w_1^*\langle 1| + \cdots + w_{N-1}^*\langle N-1|)(v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle)$$

$w_0^* v_0 \langle 0|0\rangle + \quad (\langle 0|0\rangle = 1)$

$w_0^* \langle 0|1\rangle + \cdots \quad (\langle 0|1\rangle = 0)$

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

Example: $|v\rangle = |0\rangle + 2i|1\rangle$ $|w\rangle = 3i|0\rangle + (2i+2)|1\rangle$

$$\langle w|v\rangle = -3i\cdot 1 + (-2i+2)2i = 4+i$$

Norm of a Vector

Norm of a vector:

$$|||v\rangle|| = \sqrt{\langle v|v\rangle}$$

$$\begin{aligned}\langle v|v\rangle &= v_0^* v_0 + v_1^* v_1 + \cdots + v_{N-1}^* v_{N-1} \\ &= |v_0|^2 + |v_1|^2 + \cdots + |v_{N-1}|^2\end{aligned}$$

which is always a positive real number

it is the length of the complex vector

Example: $|v\rangle = |0\rangle + 2i|1\rangle$

$$\langle v|v\rangle = |1|^2 + |2i|^2 = 5$$

$$|||v\rangle|| = \sqrt{5}$$

A Different Basis

A different orthonormal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\langle +|+\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1$$

$$\langle -|-\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{-1}{\sqrt{2}}\right|^2 = 1$$

$$\langle +|-\rangle = \left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\right) = 0$$

An orthonormal basis is complete if the number of basis elements is equal to the dimension of the complex vector space.

Example Basis Change

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express $|v\rangle = v_0|0\rangle + v_1|1\rangle$ in this basis: $|v\rangle = v_+|+\rangle + v_-|-\rangle$

$$\langle +|v\rangle = \langle +|\left(v_+|+\rangle + v_-|-\rangle\right) = v_+\langle +|+\rangle + v_-\langle +|-\rangle = v_+$$

$$\langle -|v\rangle = \langle -|\left(v_+|+\rangle + v_-|-\rangle\right) = v_+\langle -|+\rangle + v_-\langle -|-\rangle = v_-$$

So: $|v\rangle = (\langle +|v\rangle)|+\rangle + (\langle -|v\rangle)|-\rangle$

$$\langle +|v\rangle = \left(\frac{1}{\sqrt{2}}v_0\right) + \left(\frac{1}{\sqrt{2}}v_1\right) = \frac{v_0 + v_1}{\sqrt{2}}$$

$$\langle -|v\rangle = \left(\frac{1}{\sqrt{2}}v_0\right) + \left(\frac{-1}{\sqrt{2}}v_1\right) = \frac{v_0 - v_1}{\sqrt{2}}$$

$$|v\rangle = \frac{v_0 + v_1}{\sqrt{2}}|+\rangle + \frac{v_0 - v_1}{\sqrt{2}}|-\rangle$$

Explicit Basis Change

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express $|v\rangle = |0\rangle$ in this basis: $|v\rangle = v_+|+\rangle + v_-|-\rangle$

$$\langle +|0\rangle = \frac{1}{\sqrt{2}}$$

$$|v\rangle = (\langle +|v\rangle)|+\rangle + (\langle -|v\rangle)|-\rangle$$

$$\langle -|0\rangle = \frac{1}{\sqrt{2}}$$

So:

$$|v\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} \end{bmatrix}$$

Matrices

A N dimensional complex matrix M is an N by N array of complex numbers:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$M_{j,k}$ are complex numbers

Example:

Three dimensional complex matrix:

$$M = \begin{bmatrix} 4 & 3+i & 2 \\ i & e^{\frac{\pi}{4}} & \sqrt{2}i \\ 0 & 0 & 4 \end{bmatrix}$$

$M_{1,0} = i$

$M_{2,2} = 4$

Matrices, Multiplied by Scalar

Matrices can be multiplied by a complex number

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$\alpha M = \begin{bmatrix} \alpha M_{0,0} & \cdots & \alpha M_{0,N-1} \\ \vdots & & \vdots \\ \alpha M_{N-1,0} & \cdots & \alpha M_{N-1,N-1} \end{bmatrix}$$

Example: $M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix}$ $\alpha = 2i$

$$\alpha M = \begin{bmatrix} 2i \cdot 0 & 2i(3+i) \\ 2i(i) & 2i(1) \end{bmatrix} = \begin{bmatrix} 0 & -2+6i \\ -2 & 2i \end{bmatrix}$$

Matrices, Added

Matrices can be added

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} L = \begin{bmatrix} L_{0,0} & \cdots & L_{0,N-1} \\ \vdots & & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} \end{bmatrix}$$

$$M+L = \begin{bmatrix} M_{0,0}+L_{0,0} & \cdots & M_{0,N-1}+L_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0}+L_{N-1,0} & \cdots & M_{N-1,N-1}+L_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 3 & -3-i \\ i & 2 \end{bmatrix}$$

$$M+L = \begin{bmatrix} 0+3 & 3+i+(-3-i) \\ i+i & 1+2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2i & 3 \end{bmatrix}$$

Matrices, Complex Conjugate

Given a matrix, we can form its complex conjugate by conjugating every element:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^* = \begin{bmatrix} M^*_{0,0} & \cdots & M^*_{0,N-1} \\ \vdots & & \vdots \\ M^*_{N-1,0} & \cdots & M^*_{N-1,N-1} \end{bmatrix}$$

Example: $M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix}$ $M^* = \begin{bmatrix} 0 & 3-i \\ -i & 1 \end{bmatrix}$

Matrices, Transpose

Given a matrix, we can form it's transpose by reflecting across the diagonal

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^T = \begin{bmatrix} M_{0,0} & \cdots & M_{N-1,0} \\ \vdots & & \vdots \\ M_{0,N-1} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

Example: $M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix}$ $M^T = \begin{bmatrix} 0 & i \\ 3+i & 1 \end{bmatrix}$

Matrices, Conjugate Transpose

Given a matrix, we can form its conjugate transpose by reflecting across the diagonal and conjugating

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^\dagger = \begin{bmatrix} M^*_{0,0} & \cdots & M^*_{N-1,0} \\ \vdots & & \vdots \\ M^*_{0,N-1} & \cdots & M^*_{N-1,N-1} \end{bmatrix}$$

Example: $M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix}$ $M^\dagger = \begin{bmatrix} 0 & -i \\ 3-i & 1 \end{bmatrix}$