تمرین ۳

تمام «مثال‌هایی» که در باکس‌های خاکستری رنگ آمده است را به صورت مختصر توضیح دهید.

مثال: در صفحه دو تعدادی حالت ممکن و غیرممکن برای یک کیوبیت را مشاهده می‌کنیم. شرط لازم برای امکان‌پذیری یک حالت آن است که مجموع احتمالات حضور در هر حالت پایه برابر یک شود.

Qubits

Two dimensional quantum systems are called qubits

A qubit has a wave function which we write as

$$|v\rangle = v_0|0\rangle + v_1|1\rangle \qquad |v_0|^2 + |v_1|^2 = 1$$

Examples:

Valid qubit wave functions:

$$|v\rangle = |0\rangle \qquad |||v\rangle|| = \sqrt{|1|^2 + |0|^2} = 1$$

$$|v\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \qquad |||v\rangle|| = \sqrt{\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{\sqrt{2}}\right|^2} = 1$$

Invalid qubit wave function (not normalized):

$$|v\rangle = 5|0\rangle + i|1\rangle \qquad |||v\rangle|| = \sqrt{|5|^2 + |i|^2} = \sqrt{26}$$

Measuring Qubits

Example:

We are given a qubit with wave function

$$|v\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle \qquad |||v\rangle|| = \sqrt{\left|\frac{1}{\sqrt{3}}\right|^2 + \left|i\sqrt{\frac{2}{3}}\right|^2} = 1$$

If we observe the system in the computational basis, then we get outcome 0 with probability

$$\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

and we get outcome 1 with probability:

$$\left|\frac{\sqrt{2}i}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

Measuring Qubits

Example:

We are given a qubit with wave function

$$|v\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle \qquad |||v\rangle|| = \sqrt{\left|\frac{1}{\sqrt{3}}\right|^2 + \left|i\sqrt{\frac{2}{3}}\right|^2} = 1$$

If we observe the system in the computational basis, then we get outcome 0 with probability

$$\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$ new wave function $|0\rangle$

and we get outcome 1 with probability:

$$\left|\frac{2i}{\sqrt{3}}\right|^2 = \frac{2}{3}$$ new wave function $|1\rangle$

Measuring Qubits

Example:

We are given a qubit with wave function

$|v\rangle = |0\rangle$ $\qquad$ $|||v\rangle|| = \sqrt{|1|^2 + |0|^2} = 1$

If we observe the system in the computational basis, then we get outcome 0 with probability

$|1|^2 = 1$ $\qquad$ new wave function $|0\rangle$

and we get outcome 1 with probability:

$|0|^2 = 0$ $\qquad$ a.k.a never

Unitary Evolution for Qubits

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$|v'\rangle = U|v\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} \\ \frac{-i}{\sqrt{2}}\frac{1}{2} + \frac{i}{\sqrt{2}}\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{(-1+\sqrt{3})i}{2\sqrt{2}} \end{bmatrix}$$

$$|v'\rangle = \frac{1+\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{(-1+\sqrt{3})i}{2\sqrt{2}}|1\rangle$$

Two Qubits

Examples:

$$|v\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|01\rangle$$

$$|v\rangle = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

$$|v\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$

Two Qubits, Separable

Example: $|v\rangle = |a\rangle \otimes |b\rangle$

$$|a\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \qquad |b\rangle = \frac{i}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

$$|v\rangle = |a\rangle\otimes|b\rangle = \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)\otimes\left(\frac{i}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$$

$$= \frac{1}{2}\frac{i}{\sqrt{5}}|0\rangle\otimes|0\rangle + \frac{1}{2}\frac{2}{\sqrt{5}}|0\rangle\otimes|1\rangle + \frac{\sqrt{3}}{2}\frac{i}{\sqrt{5}}|1\rangle\otimes|0\rangle + \frac{\sqrt{3}}{2}\frac{2}{\sqrt{5}}|1\rangle\otimes|1\rangle$$

$$= \frac{i}{2\sqrt{5}}|00\rangle + \frac{1}{\sqrt{5}}|01\rangle + \frac{\sqrt{3}i}{2\sqrt{5}}|10\rangle + \frac{\sqrt{3}}{\sqrt{5}}|11\rangle$$

Two Qubits, Entangled

Example:

$$|v\rangle = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

Assume:

$|v\rangle = |a\rangle \otimes |b\rangle$ $\quad |a\rangle = a_0|0\rangle + a_1|1\rangle$ $\quad |b\rangle = b_0|0\rangle + b_1|1\rangle$

$$= a_0 b_0|00\rangle + a_0 b_1|01\rangle + a_1 b_0|10\rangle + a_1 b_1|11\rangle$$

$a_0 b_1 = 0$

Either

$a_0 = 0$ but this implies $a_0 b_0 = 0$

or

$b_1 = 0$ but this implies $a_1 b_1 = 0$

contradictions

So $|v\rangle$ is not a separable state. It is entangled.

Two Qubits, Measuring

Example:

$$|v\rangle = \frac{i}{2\sqrt{5}}|00\rangle + \frac{1}{\sqrt{5}}|01\rangle + \frac{\sqrt{3}i}{2\sqrt{5}}|10\rangle + \frac{\sqrt{3}}{\sqrt{5}}|11\rangle$$

Probability of 00 is $|v_{00}|^2 = \left|\frac{i}{2\sqrt{5}}\right|^2 = \frac{1}{20}$

Probability of 01 is $|v_{01}|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5}$

Probability of 10 is $|v_{10}|^2 = \left|\frac{\sqrt{3}i}{2\sqrt{6}}\right|^2 = \frac{3}{20}$

Probability of 11 is $|v_{11}|^2 = \left|\frac{\sqrt{3}}{\sqrt{5}}\right|^2 = \frac{3}{5}$

Two Qubit Evolutions

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

$$|v'\rangle = \begin{bmatrix} U_{00,00}v_{00} + U_{00,01}v_{01} + U_{00,10}v_{10} + U_{00,11}v_{11} \\ U_{01,00}v_{00} + U_{01,01}v_{01} + U_{01,10}v_{10} + U_{01,11}v_{11} \\ U_{10,00}v_{00} + U_{10,01}v_{01} + U_{10,10}v_{10} + U_{10,11}v_{11} \\ U_{11,00}v_{00} + U_{11,01}v_{01} + U_{11,10}v_{10} + U_{11,11}v_{11} \end{bmatrix}$$

$$|v'\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{1}{2} + \frac{i}{\sqrt{2}} \cdot 0 + 0 \cdot 0 + 0 \cdot \frac{\sqrt{3}}{2} \\ \frac{i}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 0 + 0 \cdot \frac{\sqrt{3}}{2} \\ 0 \cdot \frac{1}{2} + 0 \cdot 0 + 0 \cdot 0 + 1 \cdot \frac{\sqrt{3}}{2} \\ 0 \cdot \frac{1}{2} + 0 \cdot 0 + 1 \cdot 0 + 0 \cdot \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

Tensor Product of Matrices

Example:

$$U = V \otimes W$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad W = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$U = \begin{bmatrix} V_{0,0}W_{0,0} & V_{0,0}W_{0,1} & V_{0,1}W_{0,0} & V_{0,1}W_{0,1} \\ V_{0,0}W_{1,0} & V_{0,0}W_{1,1} & V_{0,1}W_{1,0} & V_{0,1}W_{1,1} \\ V_{1,0}W_{0,0} & V_{1,0}W_{0,1} & V_{1,1}W_{0,0} & V_{1,1}W_{0,1} \\ V_{1,0}W_{1,0} & V_{1,0}W_{1,1} & V_{1,1}W_{1,0} & V_{1,1}W_{1,1} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}}W_{0,0} & \frac{1}{\sqrt{2}}W_{0,1} & \frac{i}{\sqrt{2}}W_{0,0} & \frac{i}{\sqrt{2}}W_{0,1} \\ \frac{1}{\sqrt{2}}W_{1,0} & \frac{1}{\sqrt{2}}W_{1,1} & \frac{i}{\sqrt{2}}W_{1,0} & \frac{i}{\sqrt{2}}W_{1,1} \\ \frac{i}{\sqrt{2}}W_{0,0} & \frac{i}{\sqrt{2}}W_{0,1} & \frac{1}{\sqrt{2}}W_{0,0} & \frac{1}{\sqrt{2}}W_{0,1} \\ \frac{i}{\sqrt{2}}W_{1,0} & \frac{i}{\sqrt{2}}W_{1,1} & \frac{1}{\sqrt{2}}W_{1,0} & \frac{1}{\sqrt{2}}W_{1,1} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{1}{2} & \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} & \frac{i}{\sqrt{2}}\frac{1}{2} & \frac{i}{\sqrt{2}}\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}}\frac{-1}{2} & \frac{i}{\sqrt{2}}\frac{\sqrt{3}}{2} & \frac{i}{\sqrt{2}}\frac{-1}{2} \\ \frac{i}{\sqrt{2}}\frac{1}{2} & \frac{i}{\sqrt{2}}\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}}\frac{1}{2} & \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} \\ \frac{i}{\sqrt{2}}\frac{\sqrt{3}}{2} & \frac{i}{\sqrt{2}}\frac{-1}{2} & \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}}\frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{\sqrt{3}i}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{\sqrt{3}i}{2\sqrt{2}} & \frac{-i}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{\sqrt{3}i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{\sqrt{3}i}{2\sqrt{2}} & \frac{-i}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \end{bmatrix}$$

Tensor Product of Matrices

Example:

$$U = V \otimes I$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} V_{0,0}W_{0,0} & V_{0,0}W_{0,1} & V_{0,1}W_{0,0} & V_{0,1}W_{0,1} \\ V_{0,0}W_{1,0} & V_{0,0}W_{1,1} & V_{0,1}W_{1,0} & V_{0,1}W_{1,1} \\ V_{1,0}W_{0,0} & V_{1,0}W_{0,1} & V_{1,1}W_{0,0} & V_{1,1}W_{0,1} \\ V_{1,0}W_{1,0} & V_{1,0}W_{1,1} & V_{1,1}W_{1,0} & V_{1,1}W_{1,1} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}}1 & \frac{1}{\sqrt{2}}0 & \frac{i}{\sqrt{2}}1 & \frac{i}{\sqrt{2}}0 \\ \frac{1}{\sqrt{2}}0 & \frac{1}{\sqrt{2}}1 & \frac{i}{\sqrt{2}}0 & \frac{i}{\sqrt{2}}1 \\ \frac{i}{\sqrt{2}}1 & \frac{i}{\sqrt{2}}0 & \frac{1}{\sqrt{2}}1 & \frac{1}{\sqrt{2}}0 \\ \frac{i}{\sqrt{2}}0 & \frac{i}{\sqrt{2}}1 & \frac{1}{\sqrt{2}}0 & \frac{1}{\sqrt{2}}1 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Tensor Product of Matrices

Example:

$$U = I \otimes W$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = \begin{bmatrix} V_{0,0}W_{0,0} & V_{0,0}W_{0,1} & V_{0,1}W_{0,0} & V_{0,1}W_{0,1} \\ V_{0,0}W_{1,0} & V_{0,0}W_{1,1} & V_{0,1}W_{1,0} & V_{0,1}W_{1,1} \\ V_{1,0}W_{0,0} & V_{1,0}W_{0,1} & V_{1,1}W_{0,0} & V_{1,1}W_{0,1} \\ V_{1,0}W_{1,0} & V_{1,0}W_{1,1} & V_{1,1}W_{1,0} & V_{1,1}W_{1,1} \end{bmatrix}$$

$$U = \begin{bmatrix} 1\frac{1}{\sqrt{2}} & 1\frac{i}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} & 0\frac{i}{\sqrt{2}} \\ 1\frac{i}{\sqrt{2}} & 1\frac{1}{\sqrt{2}} & 0\frac{i}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} \\ 0\frac{1}{\sqrt{2}} & 0\frac{i}{\sqrt{2}} & 1\frac{1}{\sqrt{2}} & 1\frac{i}{\sqrt{2}} \\ 0\frac{i}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} & 1\frac{i}{\sqrt{2}} & 1\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Tensor Product of Matrices

Example:

$$U = I \otimes W$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = \begin{bmatrix} V_{0,0}W_{0,0} & V_{0,0}W_{0,1} & V_{0,1}W_{0,0} & V_{0,1}W_{0,1} \\ V_{0,0}W_{1,0} & V_{0,0}W_{1,1} & V_{0,1}W_{1,0} & V_{0,1}W_{1,1} \\ V_{1,0}W_{0,0} & V_{1,0}W_{0,1} & V_{1,1}W_{0,0} & V_{1,1}W_{0,1} \\ V_{1,0}W_{1,0} & V_{1,0}W_{1,1} & V_{1,1}W_{1,0} & V_{1,1}W_{1,1} \end{bmatrix}$$

$$U = \begin{bmatrix} 1\frac{1}{\sqrt{2}} & 1\frac{i}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} & 0\frac{i}{\sqrt{2}} \\ 1\frac{i}{\sqrt{2}} & 1\frac{1}{\sqrt{2}} & 0\frac{i}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} \\ 0\frac{1}{\sqrt{2}} & 0\frac{i}{\sqrt{2}} & 1\frac{1}{\sqrt{2}} & 1\frac{i}{\sqrt{2}} \\ 0\frac{i}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} & 1\frac{i}{\sqrt{2}} & 1\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Linearity

Example:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|v\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

$$|v'\rangle = U|v\rangle = \frac{1}{2}U|00\rangle + \frac{\sqrt{3}}{2}U|11\rangle$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|01\rangle\right) + \frac{\sqrt{3}}{2}|10\rangle$$

$$= \frac{1}{2\sqrt{2}}|00\rangle + \frac{i}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2}|10\rangle$$

Linearity

Example:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|v\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|01\rangle$$

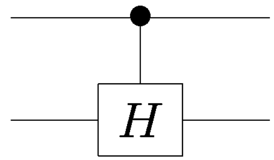
$$|v'\rangle = U|v\rangle = \frac{1}{2}U|00\rangle + \frac{\sqrt{3}}{2}U|01\rangle$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|01\rangle\right) + \frac{\sqrt{3}}{2}\left(\frac{i}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle\right)$$

$$= \frac{1 + i\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3} + i}{2\sqrt{2}}|01\rangle$$

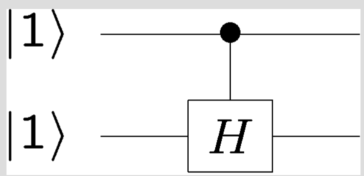
Quantum Circuits

controlled-H



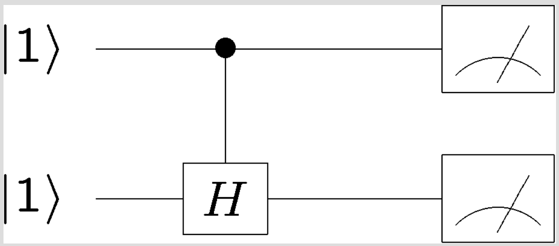
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$



$$|v'\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$



Probability of 10: $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

Probability of 11: $\left|\frac{-1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

Probability of 00 and 01: $|0|^2 = 0$

Matrices, Bras, and Kets

So far we have used bras and kets to describe row and column vectors. We can also use them to describe matrices:

Outer product of two vectors:

$$|v\rangle\langle w| = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} w_1^* & w_2^* \end{bmatrix} = \begin{bmatrix} v_1 w_1^* & v_1 w_2^* \\ v_2 w_1^* & v_2 w_2^* \end{bmatrix}$$

Example: $|v\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ $\langle w| = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$|v\rangle\langle w| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{i}{2} & -\frac{1}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{bmatrix}$$

Matrices, Bras, and Kets

We can expand a matrix about all of the computational basis outer products

$$M = \sum_{i,j=0}^{N-1} M_{i,j} |i\rangle\langle j| = \begin{bmatrix} M_{0,0} & \cdots & M_{N-1,0} \\ \vdots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

Example:

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \qquad |1\rangle\langle 0| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = |0\rangle\langle 0| + i|0\rangle\langle 1| - 1|1\rangle\langle 0| - i|1\rangle\langle 1|$$

Matrices, Bras, and Kets

Example:

$$M = \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$$

$$M = |0\rangle\langle 0| + i|0\rangle\langle 1| - 1|1\rangle\langle 0| - i|1\rangle\langle 1|$$

$$|v\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$M|v\rangle = (|0\rangle\langle 0| + i|0\rangle\langle 1| - 1|1\rangle\langle 0| - i|1\rangle\langle 1|)\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)$$

$$= \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle - i\frac{\sqrt{3}}{2}|1\rangle$$

$$= \frac{1+i\sqrt{3}}{2}|0\rangle - \frac{1+i\sqrt{3}}{2}|1\rangle$$

M $|v\rangle$

$\frac{1}{2}|0\rangle\langle 0|0\rangle$ 1

$|0\rangle\langle 0| \frac{\sqrt{3}}{2}|1\rangle$

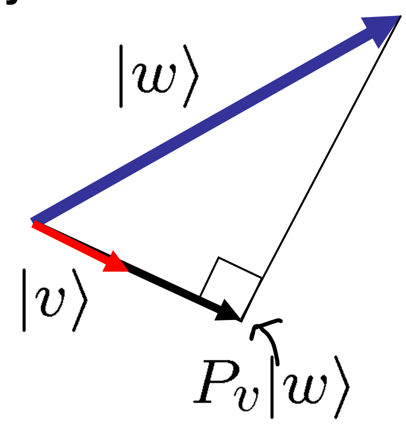
$\frac{\sqrt{3}}{2}|0\rangle\langle 0|1\rangle$ 0

Projectors

The projector onto a state $|v\rangle$ (which is of unit norm) is given by

$$P_v = |v\rangle\langle v| \qquad \langle v|v\rangle = 1$$

Projects onto the state:



Note that

$$P_v|v\rangle = |v\rangle\langle v|v\rangle = |v\rangle$$

and that

$$P_v|w\rangle = \underbrace{|v\rangle\langle v|}_{}w\rangle = (\underbrace{\langle v|w\rangle}_{\mathbb{C}})|v\rangle$$

Example: $|v\rangle = |0\rangle \qquad P_v = |0\rangle\langle 0|$

$$|w\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$P_v|w\rangle = |0\rangle\langle 0|\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) = \frac{1}{2}|0\rangle$$

Measurement Rule

If we measure a quantum system whose wave function is $|v\rangle$ in the basis $|w_i\rangle$, then the probability of getting the outcome corresponding to $|w_i\rangle$ is given by

Comp basis $|w_i\rangle \to |0\rangle, |1\rangle$ .... $|N-1\rangle$

$$Pr(|w_i\rangle) = |\langle w_i|v\rangle|^2 = \langle v|w_i\rangle\langle w_i|v\rangle = \langle v|P_{w_i}|v\rangle$$

where

$$P_{w_i} = |w_i\rangle\langle w_i|$$

The new wave function of the system after getting the measurement outcome corresponding to $|w_i\rangle$ is given by

$$|v'\rangle = \frac{P_{w_i}|v\rangle}{\sqrt{Pr(|w_i\rangle)}}$$

For measuring in a complete basis, this reduces to our normal prescription for quantum measurement, but…

Measuring One of Two Qubits

Suppose we measure the first of two qubits in the computational basis. Then we can form the two projectors:

$$P_0 \otimes I = |0\rangle\langle 0| \otimes I \qquad I = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$P_1 \otimes I = |1\rangle\langle 1| \otimes I \qquad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If the two qubit wave function is $|v\rangle$ then the probabilities of these two outcomes are

$$Pr(0) = \langle v|P_0 \otimes I|v\rangle$$

$$Pr(1) = \langle v|P_1 \otimes I|v\rangle$$

And the new state of the system is given by either

$$|v'\rangle = \frac{P_0 \otimes I|v\rangle}{\sqrt{Pr(0)}}$$

Outcome was 0

$$|v'\rangle = \frac{P_1 \otimes I|v\rangle}{\sqrt{Pr(1)}}$$

Outcome was 1

Measuring One of Two Qubits

Example: $|v\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measure the first qubit: $P_0 \otimes I = |0\rangle\langle 0| \otimes I$ $P_1 \otimes I = |1\rangle\langle 1| \otimes I$

$$P_0 \otimes I = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |00\rangle\langle 00| + |01\rangle\langle 01|$$

$$Pr(0) = \langle v|P_0 \otimes I|v\rangle$$

$$P_0 \otimes I|v\rangle = (|00\rangle\langle 00| + |01\rangle\langle 01|)|v\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle$$

$$Pr(0) = \left(\frac{1}{2}\langle 00| + \frac{1}{2}\langle 01| + \frac{1}{\sqrt{2}}\langle 11|\right)\left(\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle\right)$$

$$Pr(0) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$|v'\rangle = \frac{P_0 \otimes I|v\rangle}{\sqrt{Pr(0)}} = \frac{\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle}{\frac{1}{\sqrt{2}}}$$

$$|v'\rangle = \frac{1}{\sqrt{2}}|00\rangle + \sqrt{\frac{1}{2}}|01\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \sqrt{\frac{1}{2}}|1\rangle\right)$$