Quantum Information Processing

Lecture 3

Postulates

Complex numbers ($i^2 = -1$)

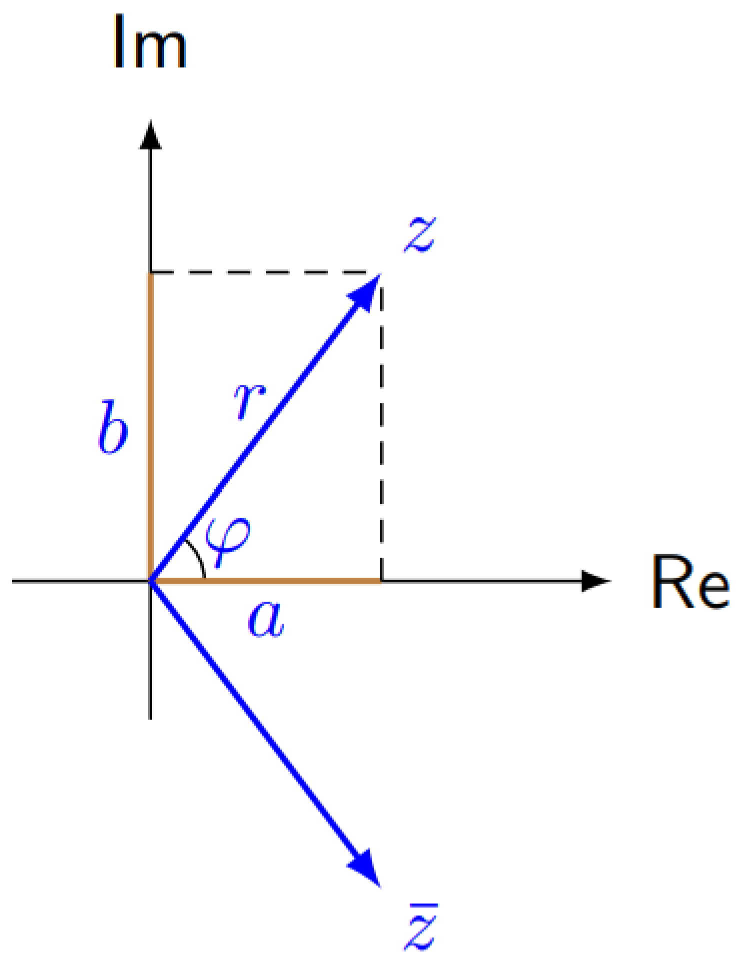
Representations:

- algebraic: $z = a + ib$
- exponential: $z = re^{i\phi} = r(\cos\phi + i\sin\phi)$

Operations:

- addition and subtraction:
  $$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$
- multiplication:
  $$(a + ib) . (c + id) = (ac - bd) + i(ad + bc)$$
  $$re^{i\phi}.r'e^{i\phi'} = rr'e^{i(\phi+\phi')}$$
- complex conjugate:
  $$z^* = \bar{z} = a - ib = re^{-i\phi}$$
- Absolute value:
  $$|z| = \sqrt{a^2 + b^2} = r, \ |z_1.z_2| = |z_1|.|z_2|$$
- absolute value squared: $|z|^2 = a^2 + b^2 = r^2$

  important: $|z|^2 = z\bar{z}$
- inverse: $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$



Global and relative phases

Phase

If $re^{i\phi}$ is a complex number, $e^{i\phi}$ is called phase.

Global phase

The following states differ only by a global phase:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad e^{i\phi}|\psi\rangle = e^{i\phi}\alpha|0\rangle + e^{i\phi}\beta|1\rangle$$

These states are indistinguishable! Why? Because $|\alpha|^2 = |e^{i\phi}\alpha|^2$ and $|\beta|^2 = |e^{i\phi}\beta|^2$ so it makes no difference during measurements.

Relative phase

These states differ by a relative phase:

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Are they also indistinguishable? No! (Measure in a *different basis*.)

Remember: global phase does not matter, relative phase matters!

Remember that:

$$e^{i\phi} = 1 \times e^{i\phi} \rightarrow r = 1$$
$$\left|e^{i\phi}\right| = \sqrt{1^2} = 1$$
$$\left|e^{i\phi}\alpha\right| = \left|e^{i\phi}\right|.\,|\alpha| = |\alpha|$$

Qubit states: the Bloch sphere

Any qubit state can be written as

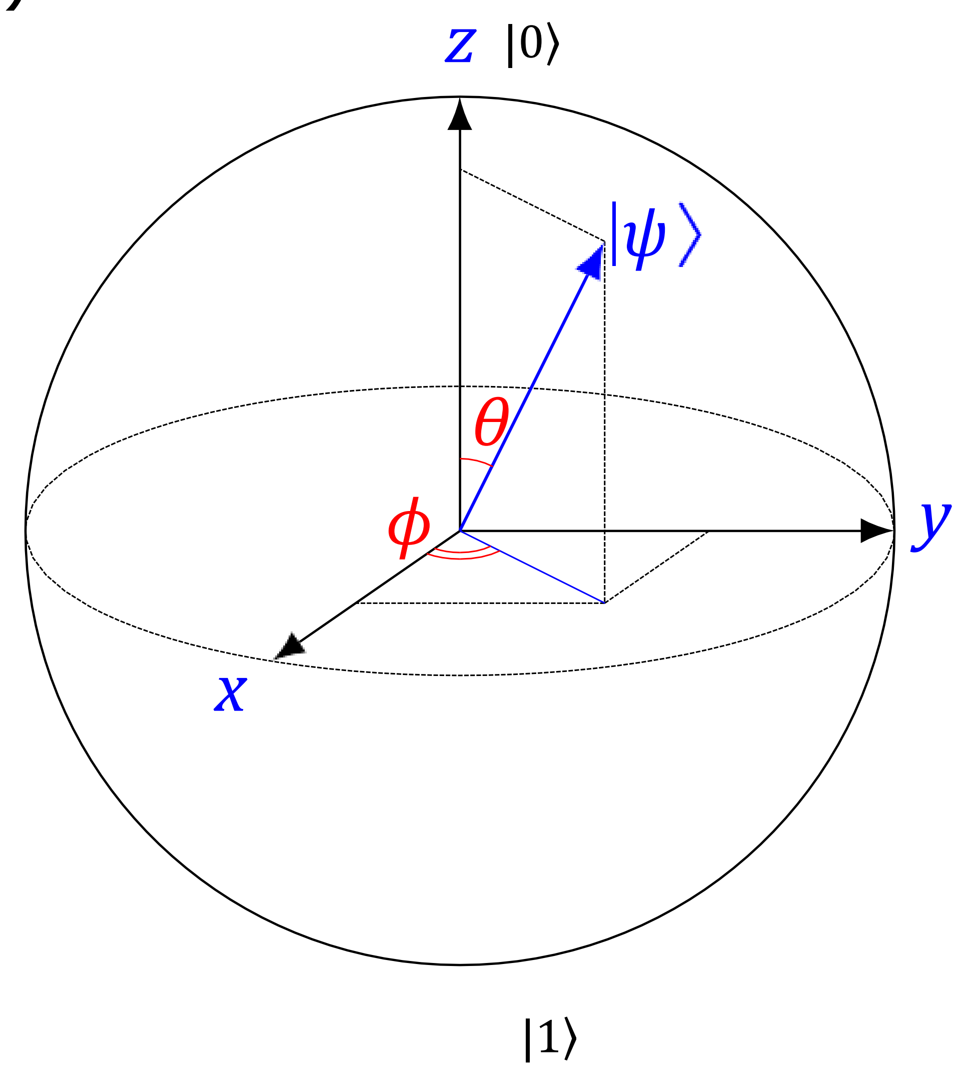
$$|\psi\rangle = \underbrace{\cos\frac{\theta}{2}}_{\alpha}|0\rangle + \underbrace{e^{i\varphi}\sin\frac{\theta}{2}}_{\beta}|1\rangle$$

for some angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

There is a one-to-one correspondence between qubit states and points on a unit sphere (also called Bloch sphere):

Bloch vector of $|\psi\rangle$ in spherical coordinates:

$$\begin{cases} x = \sin\theta\cos\phi \\ y = \sin\theta\sin\phi \\ z = \cos\theta \end{cases}$$



Measurement probabilities:

$$|\alpha|^2 = \left(\cos\frac{\theta}{2}\right)^2 = \frac{1}{2} + \frac{1}{2}\cos\theta$$

$$|\beta|^2 = \left(\sin\frac{\theta}{2}\right)^2 = \frac{1}{2} - \frac{1}{2}\cos\theta$$

It might, at first sight, seem that there should be four degrees of freedom in $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, as α and β are complex numbers with two degrees of freedom each.
However, one degree of freedom is removed by the normalization constraint $\alpha^2 + \beta^2 = 1$. This means, with a suitable change of coordinates, one can eliminate one of the degrees of freedom. One possible choice is that of Hopf coordinates:

$$\alpha = e^{i\psi} \cos\frac{\theta}{2}$$

$$\beta = e^{i(\psi+\phi)} \sin\frac{\theta}{2}.$$

Additionally, for a single qubit the overall phase of the state $e^{i\psi}$ has no physically observable consequences, so we can arbitrarily choose α to be real (or β in the case that α is zero), leaving just two degrees of freedom:

$$\alpha = \cos\frac{\theta}{2},$$

$$\beta = e^{i\phi} \sin\frac{\theta}{2},$$

where $e^{i\phi}$ is the physically significant relative phase.